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The limits of logic

by David Guaspari

A review of *Infinitesimal: How a Dangerous Mathematical Theory Shaped the Modern World* by Amir Alexander

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Books in this article

Amir Alexander

Infinitesimal: How a Dangerous Mathematical Theory Shaped the Modern World

Scientific American / Farrar, Straus and Giroux, 368 pages, $27.00

Cards on the table: I approached Amir Alexander's book warily. The title places it in a suspect genre; according to an Amazon search, our overdetermined world has been decisively shaped by, among many other things, codfish, cotton, Islam, the Scots, air conditioning, the Lunar Society of Birmingham, and, if we include the testimony of documentary films, beer. And the book's introduction suggests a morality play that flatters all the prejudices of the right-thinking: "On the one side were the advocates of intellectual freedom, scientific progress, and political reform; on the other, the champions of authority, universal and unchanging knowledge, and fixed political hierarchy."

Well and clearly written, *Infinitesimal* tells two stories about a seventeenth-century mathematical dispute. The protagonists of each are developing and applying a technique ("the method of indivisibles") that promises novel methods of discovery and proof. That technique is also rife with paradoxes and contradictions, and leads easily to errors; its advocates can give no rigorous account of what they're doing, and some dismiss the need for one. Their antagonists uphold the rigorous ideals of classical mathematics, modeled on Euclid. For them, Alexander says, these ideals are a bulwark against chaos and disorder, not only in mathematics but in thought and politics. As a result, he argues, much more was at stake than mathematical technicalities.

The antagonist of the first story is the Jesuit order, trying to restore the unity and authority of the Roman Church during the turmoil of the Reformation and Counter-Reformation. The second describes a polemical war between the philosopher (and pretty good mathematician) Thomas Hobbes and the English mathematician John Wallis that lasted from 1656 until Hobbes's death in 1679. The two men, says Alexander, responded to the anarchy of the English Civil War and Interregnum in opposite ways. Hobbes sought peace and stability by proposing classical mathematical methods as a model for and an instrument of politics; Wallis did so by rejecting the ideal of certainty. (I'm willing to accept Alexander's account of Hobbes's motives, though scholars disagree on just what role mathematics played in Hobbes's notion of "method"—of how to achieve reliable knowledge.)

The method of indivisibles comprises rules of thumb—to the adventurous, a double-oh license—for performing infinitely many operations on infinitely small things and managing to arrive at some finite, and meaningful, result. "Indivisible" is the usual term for an infinitely small geometrical object and "infinitesimal" for an infinitely small (but nonzero) number—assuming such things exist.

Consider a squared-up deck of cards. Push it askew, so that its shape changes from a rectangular box to one with slanted sides; looked at end-on, its profile is not a rectangle but (approximately) a parallelogram. The sides of the squared-up deck are smooth planes, but the sides of the skewed stack are like staircases with steps as high as the cards are thick. The new shape occupies the same volume as the old one, since both are composed of the same pieces. If a deck consisted of twice as many cards, each half as thick as a card from the original, the sides of the skewed deck would be smoother. Now for the leap: Suppose it makes sense to say that a deck is made of infinitely many infinitely thin cards. Then, it seems, the sides of this skewed deck would be smooth planes; and we might still, hopefully, conclude that it has the same volume as the squared-up shape we started with. This reasoning, if valid, should also apply to a more complex shape—e.g., one whose profile, looked at end-on, is like a thick letter "S." The problem of calculating the volume of this complex, wavy figure would be reduced to calculating the volume of a rectangular box.

Disquieting problems arise if we try using indivisibles to compute the volume occupied by the squared-up deck. The method assumes it is the sum of the volumes of all the infinitely thin cards. What is the volume of one such card? If zero, we seem forced to conclude that the volume of the deck is zero, because it's a sum of zeroes. If not, we conclude that the volume of the deck is infinite, the result of incrementing by the same positive number infinitely often. Difficulties like this can be multiplied endlessly. It seems hard to make sense of the procedure, and philosophical arguments going back at least as far as Aristotle say that it is not possible to do so.

And yet . . . those guys were onto something. Bold and imaginative use of indivisibles and infinitesimals solved old problems and led to new results—and could also establish known results in ways that, it was argued, gave more insight into why they were true. Evangelista Torricelli, Galileo's successor as court mathematician to the Medici, published twenty-one different proofs of a result of Archimedes—eleven by standard means, ten by indivisibles—to provide both a virtuoso demonstration of how to use indivisibles and, he argued, a demonstration of their power.

Bonaventura Cavalieri, who published the standard treatise on indivisibles in 1635, saw a need to defend them philosophically, but his arguments were obscure. Many able mathematicians, such as Fermat, took a middle ground: the methods could be helpful heuristic guides, but their results would only be plausible conjectures, not theorems, until proven by wellestablished methods. Wallis was an extremist. He asserted that Cavalieri had answered all philosophical objections and that mathematics was, in any case, a kind of experimental science. One could, for example, establish a general result by "induction" from a set of examples: checking that a result holds true for several numbers and noting some emerging pattern would allow one to conclude, with an acceptably low likelihood of error, that it's true for all numbers. Fermat gagged. (Alexander says, in an odd aside, that although famous and esteemed in his own time, Fermat "is remembered today mostly as the author of Fermat's Last Theorem"—as though he were some reactionary now swept into the dustbin of history. Nowadays, every mathematics undergraduate studies Fermat's work, and he has always been in the pantheon.)

The Jesuits effectively stamped out the teaching of infinitesimal methods in Italy, but the method flourished elsewhere, opening the door to Newton's development of calculus (ca. 1665) and his mathematical physics, and therefore to the flourishing of modern science. Alexander ends his stories at the threshold of Newton's achievements, but later plot twists are worth noting. One triumph of nineteenth-century mathematics was to provide a rigorous foundation for the achievements of the two preceding centuries. It did so by banishing infinitesimals. Now they were needed not by the cutting-edge guys but by the squares, such as the teachers Bertrand Russell complained about in his autobiography: "My mathematical tutors had never shown me any reason to suppose the Calculus anything but a tissue of fallacies."

In modern mathematical practice, rigor is not a contested notion, and the agreed- upon standard could not, even in principle, become any more demanding than it is. It is fascinating to look back at a brilliant era in mathematics when that was not so and when, hindsight suggests, the way forward was to work without a net.

Alexander's account of the mathematics should be accessible to anyone willing to make a modest effort. From it I gratefully learned new things—e.g., how Torricelli *cultivated* paradox and could exploit it. One paradox, he suggested, showed that different infinitely thin slices must have different thicknesses and even implied a way to calculate the ratios between them. Then he used that calculation to get new results. His bravado reminded me of what a colleague once said about a proof by someone who played the game at a much higher level than either of us: Who else would have had the courage to try *that*?

In Alexander's telling, the Hobbes/Wallis dispute, for all its bitterness, seems little more than a kind of nasty comedy. As a practical matter, neither was in a position to enforce his views, and, as an intellectual matter, the views of both were so idiosyncratic it seems hard to see what larger lesson to draw from their conflict. According to Alexander, Wallis believed that an attachment to order and rigor was actually incompatible with freedom, progress, and social comity, because classical mathematical methods limited free debate by claiming certainty for their results—which makes him sound like a parody of a modern educational faddist who believes that every child should have his own value of 2+2.

The conflict between Jesuits and "indivisibilists" was another matter. The Jesuits had prestige and influence, and their view of the world was a live option. By the end of the sixteenth century the advocacy of Christopher Clavius, an able mathematician, had persuaded his Jesuit superiors to make mathematics a central study in their spectacular educational system. Euclidean geometry provided a powerful model of certain knowledge, a notion congenial to their Counter-Reformation project of reunifying Europe under the Church. And starting in 1601, and continuing for at least fifty years, the governing body for Jesuit colleges repeatedly issued bans on teaching indivisibles. Alexander details the Vatican politics and academic politics through which they tried to effect that ban in places not under their direct control. In Italy, he says, they essentially succeeded, a rich tradition of Italian mathematics withered, and the modern world passed Italy by.

Controversies about infinitesimals mattered, Alexander says, because serious men believed that they touched on important truths about the nonmathematical world. Infinitesimal methods ultimately undermined the "dream" that "the world is a perfectly rational place, governed by strict mathematical rules." They "demonstrate[d] that reality can never be reduced to strict mathematical reasoning" and thereby "liberated the social and political order from the need for inflexible political hierarchies."

How well does Alexander justify this cadenza, either as an insight into the beliefs and motives of the participants or as a philosophical thesis about the relations of mathematics, science, and society? He succeeds in showing that, for the Jesuit authorities, much was at stake, and banning a practice that threatened to undermine the very model of certain knowledge made perfect sense. (He attempts to present participants from their own point of view, so it's a jolt to read, twice, his assertion of a borderline-malicious cliché: the Church taught that God's "grace was channeled exclusively through ordained priests endowed with special powers." That was not taught by the Council of Trent, which kicked off the Counter-Reformation, and has never been Catholic doctrine.)

As a general thesis, the logic or association of ideas that Alexander describes is less plausible. It did not seem compelling to some important participants, including advocates of indivisibles. Galileo, for example, did not reject a universe governed by mathematical rules; he famously said that "the book of nature is written in the language of mathematics." And Leibniz, who used infinitesimals to formulate his version of calculus, was a supreme rationalist for whom, as Roger Scruton says, "the structure of reality conforms to the principles of rational argument." Further, since modern mathematics eliminates any need to appeal to infinitesimals, their use in the seventeenth and eighteenth centuries cannot be said to "demonstrate" anything about the relation between mathematics and reality.

I would recommend this book as a vivid account of some remarkable intellectual history, while adding a caution about its most sweeping claims: trust, but verify.

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